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# Effective moduli of an anisotropic material with elliptical holes of arbitrary orientational distribution

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**Abstract**

Effective moduli of a two-dimensional anisotropic solid with elliptical holes having an arbitrary (non-random) orientational distribution are given in closed form. The results are derived in the non-interacting approximation. Besides being rigorous at small defect densities, this approximation constitutes the basic building block for various approximate schemes. Proper tensorial parameters of defect density (dependent on ellipses' eccentricity and their orientations *relative* to the matrix anisotropy axes) are identified. When derived in terms of such parameters, expressions for the effective moduli cover, in a unified way, *mixtures* of holes of diverse eccentricities and arbitrary orientational distribution. A number of special cases (circles, cracks of various orientational distributions) are discussed. If the field of defects is "geometrically isotropic" (holes of the circular shapes, or randomly oriented cracks), it reduces the matrix anisotropy. 

*Keywords:* Anisotropy; Holes; Cracks; Effective moduli

## 1. Introduction and basic equations

The overall elastic properties of a two-dimensional anisotropic matrix with elliptical holes having arbitrary orientational distribution (Fig. 1) are analyzed. Results are obtained for the general matrix anisotropy, with particular attention to the case when the matrix is orthotropic. Some preliminary results for orthotropic matrix have been presented by Tsukrov and Kachanov (1998).

One of the key problems is to identify the proper parameters of hole density. We call the density parameter *proper* if it correctly reflects the individual hole contributions to the overall elastic compliance. Only in terms of such parameters can the effective compliances be uniquely expressed.

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When derived in terms of proper parameters, expressions for the effective moduli cover, in a unified way, *mixtures* of holes of diverse eccentricities and arbitrary orientational distribution.

Identification of the proper density parameters (generally, tensorial) is a non-trivial problem, since the individual hole contributions into the overall compliances depend not only on the hole shapes but on their orientations *relative* to the matrix anisotropy axes. We show, following Kachanov et al. (1994), that such parameters are implied by the structure of the elastic potential.

For illustration, consider a two-dimensional solid with cracks. A scalar crack density parameter (for randomly oriented cracks) was introduced by Bristow (1960):

$$\rho = \frac{1}{A}\sum l^{(k)2} \tag{1}$$

where $A$ is the representative area, $2l^{(k)}$ is the length of $k$th crack). For non-random crack orientations, $\rho$ was generalized by Kachanov (1980) to second rank crack density tensor

$$\boldsymbol{\alpha} = \frac{1}{A}\sum (l^2\mathbf{nn})^{(k)} \tag{2}$$

where $\mathbf{n}$ is a unit normal to crack and $\mathbf{nn}$ denotes a dyadic product with components $n_i n_j$.

In the case of cracks in the *isotropic* matrix, expressing the effective moduli in terms of $\boldsymbol{\alpha}$ covers all orientational distributions in a unified way and establishes the overall anisotropy. However, in the case of an *anisotropic* matrix, $\boldsymbol{\alpha}$ (or $\rho$ in the case of random orientations) cannot always be used as density parameter, for the following reasons.

Tensor $\boldsymbol{\alpha}$ and scalar $\rho$ take the individual crack contribution as proportional to $l^2$, whereas the actual "relative weight" of an individual crack should be adjusted according to the orientation *with respect to the matrix anisotropy axes*. (Cracks normal to the stiffer direction of the matrix produce a higher impact on the overall compliance, as compared to cracks normal to the "softer" direction). As shown by Mauge and Kachanov (1994) and in the text to follow, these considerations give rise to fourth rank tensor of crack density (that can be replaced, in the case of the isotropic matrix, by second rank crack density tensor $\boldsymbol{\alpha}$).

Similarly, in the case of holes, the usual density parameter — relative area of holes (porosity) $p$ that takes the individual hole contributions as proportional to their areas — is adequate only in the case of *circular* holes. For non-circular holes, the individual hole contributions depend on the hole shapes and orientations; the proper hole density parameter should reflect these factors (see Kachanov et al., 1994).

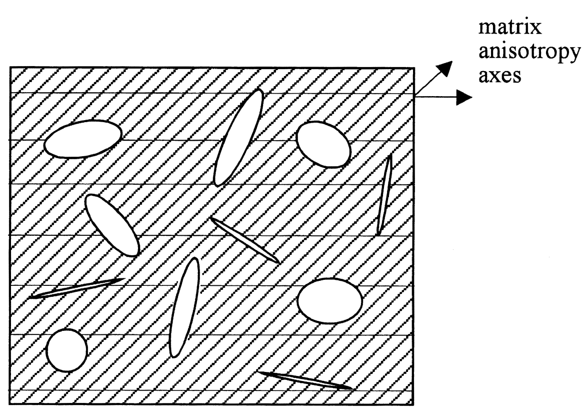


Fig. 1. Anisotropic matrix with arbitrarily oriented elliptical holes.

The present work incorporates both anisotropy of the matrix and non-circularity of holes, by considering elliptical holes arbitrarily oriented in an anisotropic matrix. Besides providing a unified treatment, this approach has the advantage of covering *mixtures* of defects of diverse shapes (including cracks).

We start with the analysis of one isolated elliptical hole, arbitrarily oriented in an anisotropic matrix. The overall strain (per reference area $A$) in a material containing a hole with boundary $\Gamma$ having a unit normal $\mathbf{n}$ (directed outward to material, i.e. inside the hole) under remotely applied stress $\boldsymbol{\sigma}$ is

$$\boldsymbol{\varepsilon} = \mathbf{S}^0 : \boldsymbol{\sigma} + \Delta\boldsymbol{\varepsilon} \quad \left(\text{or } \varepsilon_{ij} = S^0_{ijkl}\sigma_{kl} + \Delta\varepsilon_{ij}, \text{ in indicial notations}\right) \tag{3}$$

where $\mathbf{S}^0$ is the compliance tensor of the matrix and a colon denotes contraction over two indices. The strain due to cavity

$$\Delta\boldsymbol{\varepsilon} = -\frac{1}{2A}\int_\Gamma (\mathbf{un} + \mathbf{nu})\,\mathrm{d}\Gamma \tag{4}$$

where $\mathbf{u}$ denotes displacements of the points of $\Gamma$ and $\mathbf{un}$, $\mathbf{nu}$ are dyadic (tensor) products of two vectors.

Formula (4) results from application of the divergence theorem to a solid with a cavity. It is an immediate consequence of a footnote remark of Hill (1963) and was used in the explicit form by a number of authors, see, for example, Vavakin and Salganik (1975).

Due to linear elasticity, $\Delta\boldsymbol{\varepsilon}$ is a linear function of $\boldsymbol{\sigma}$ and hence can be written as

$$\Delta\boldsymbol{\varepsilon} = \mathbf{H} : \boldsymbol{\sigma} \tag{5}$$

where fourth rank tensor $\mathbf{H}$ is the *cavity compliance tensor* (possessing the usual symmetries $H_{ijkl} = H_{jilk} = H_{klij}$ implied by symmetries $\varepsilon_{ij} = \varepsilon_{ji}$, $\sigma_{ij} = \sigma_{ji}$ and by the existence of elastic potential). $\mathbf{H}$-tensors were previously found for a number of two- and three-dimensional shapes of cavities in the isotropic matrix by Tsukrov and Kachanov (1993) and Kachanov et al. (1994) and, for an anisotropic matrix with *cracks* of arbitrary orientations, by Mauge and Kachanov (1994). The present work considers elliptical holes arbitrarily oriented in an anisotropic matrix.

As seen in the text to follow, it is advantageous to formulate the problem of effective properties in terms of the elastic potential (rather than compliances): the structure of the potential implies the proper parameters of defect density.

The elastic potential in stresses $f(\boldsymbol{\sigma}) = (1/2)\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ of a solid with a hole has the form

$$f(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} : \mathbf{S}^0 : \boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\sigma} : \mathbf{H} : \boldsymbol{\sigma} \equiv f_0 + \Delta f \tag{6}$$

For the *isotropic* matrix, $f_0 = (1/2E_0)[(1+\nu_0)tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) - \nu_0(tr\,\boldsymbol{\sigma})^2$, where $E_0$, $\nu_0$ are Young's modulus and Poisson's ratio of the matrix in the 2D and 3D cases of plane stress; in plane strain, $E_0$ and $\nu_0$ are to be understood as $E_0 = E_0/(1-\nu_0^2)$ and $\nu_0 = \nu_0/(1-\nu_0)$. For the orthotropic 2D solid, $f_0$ is given by formula (19) of the next section.

In the important case of a crack, $\Gamma$ shrinks to a line and integral in (4) reduces to integration of $-\mathbf{n}(\mathbf{u}^+ - \mathbf{u}^-) - (\mathbf{u}^+ - \mathbf{u}^-)\mathbf{n}$ along the crack line, where $\mathbf{u}^+ - \mathbf{u}^-$ is the displacement discontinuity vector along the crack. For a rectilinear crack of length $2l$, unit normal $\mathbf{n}$ is constant along the crack and the integral reduces to $-(\mathbf{nb} + \mathbf{bn})$ multiplied by $l^2$. Here dimensionless vector $\mathbf{b} = \langle\mathbf{u}^+ - \mathbf{u}^-\rangle/l$ is the average over the crack displacement discontinuity normalized to $l$. Thus, the strain due to a 2D rectilinear crack (per reference area $A$) is $\Delta\boldsymbol{\varepsilon} = (l^2/A)(\mathbf{bn} + \mathbf{nb})$.

Due to linear elasticity, vector $\mathbf{b}$ is a linear function of the traction $\mathbf{n}\cdot\boldsymbol{\sigma}$ induced by $\boldsymbol{\sigma}$ at the crack site in a continuous material:

$$\mathbf{b} = \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{B} \tag{7}$$

where second rank tensor **B**, introduced by Kachanov (1992), can be called the *crack compliance tensor*. It is symmetric (as follows from Betti's reciprocity theorem, applied to the cases of normal and shear loading of the crack). **B**-tensors for several types of cracks in the isotropic matrix (elliptical cracks, cracks constrained against the normal opening, fluid filled cracks) were derived by Kachanov (1992, 1993) and for cracks arbitrarily oriented in a 2D anisotropic matrix — by Mauge and Kachanov (1992, 1994).

Thus, in the case of a 2D rectilinear crack,

$$\Delta f = \frac{1}{2}\boldsymbol{\sigma} : \Delta\boldsymbol{\varepsilon} = \frac{1}{A}\boldsymbol{\sigma} : l^2\mathbf{nBn} : \boldsymbol{\sigma} \tag{8}$$

hence identifying **H**-tensor of a crack as

$$\mathbf{H} = \frac{2l^2}{A}\mathbf{nBn} \tag{9}$$

Note that the appropriate symmetrization $H_{ijkl} = H_{jilk} = H_{klij}$ (due to $\varepsilon_{ij} = \varepsilon_{ji}$, $\sigma_{kl} = \sigma_{lk}$ and to the existence of elastic potential) implies that components of (9) are $H_{ijkl} = (l^2/2)(n_iB_{jk}n_l + n_jB_{ik}n_l + n_iB_{jl}n_k + n_jB_{il}n_k)$.

We now consider a matrix with *many* holes in the framework of *non-interaction approximation*. In this approximation, each hole is placed into $\boldsymbol{\sigma}$-field and does not experience any influence of neighbors. This approximation is of the fundamental importance, by the following reasons.

1. The results are rigorous at small defect densities, provided the mutual positions of defects are random (the last restriction is necessary, to rule out the arrangements having small overall density but consisting of widely spaced clusters of closely spaced strongly interacting defects).
2. It constitutes the basic building block for various approximate schemes (self-consistent, differential, Mori-Tanaka's) that place non-interacting defects into some sort of effective environment (effective matrix or effective stress).
3. For *cracks*, the non-interaction approximation appears to yield accurate results at high crack densities and strong interactions, provided the mutual positions of cracks are uncorrelated, due to cancellation of the competing interaction effects of shielding and amplification (see Kachanov (1992) and Mauge and Kachanov (1994) for computer experiments on interacting cracks; recent physical experiments of Carvalho and Labuz (1996), although not fully sufficient, seem to confirm this fact).

**Remark.** The non-interaction approximation should be distinguished from the small density ("dilute") limit, in which the results are linearized with respect to defect density (a typical result for a certain effective modulus M in the non-interaction approximation has the structure $M/M^0 = (1 + C\xi)^{-1}$ where $\xi$ is the appropriate density parameter and C is some constant; in the "dilute limit", it is linearized to $1 - C\xi$. This linearization is an unnecessary operation — it only reduces the range of applicability of the non-interaction approximation.

Thus, in the non-interaction approximation, the potential change due to cavities is

$$\Delta f = \frac{1}{2}\boldsymbol{\sigma} : \sum \mathbf{H}^{(k)} : \boldsymbol{\sigma} \tag{10}$$

Tensor $\mathbf{H}^* = \sum \mathbf{H}^{(k)}$ (where summation may be replaced by integration over orientations, for the computational convenience) takes the individual cavity contributions with correct "relative weights" and, thus, constitutes the proper general parameter of defect density. The text to follow identifies this parameter for elliptical holes in an anisotropic matrix and further specializes it for the cases of cracks and circles. The effective elastic moduli are obtained from

$$\varepsilon_{ij} = \frac{\partial f}{\partial \sigma_{ij}} \equiv \frac{\partial}{\partial \sigma_{ij}}\left(f_0 + \frac{1}{2}\boldsymbol{\sigma} : \mathbf{H}^* : \boldsymbol{\sigma}\right) \tag{11}$$

In the case of *cracks*, (10) takes the form

$$\Delta f = \boldsymbol{\sigma} : \left(\frac{1}{A}\right)\sum (l^2\mathbf{nBn})^{(k)} : \boldsymbol{\sigma} \tag{12}$$

This representation identifies the fourth rank tensor $(1/A)\sum(l^2\mathbf{nBn})^{(k)}$, appropriately symmetrized, as the proper general parameter of *crack density*. In the case of the *isotropic* matrix, $\mathbf{B} = (\pi/E)\mathbf{I}$, where $\mathbf{I}$ is the unit tensor. Then, since $\boldsymbol{\sigma} : \mathbf{nIn} : \boldsymbol{\sigma} = \boldsymbol{\sigma}\cdot\boldsymbol{\sigma} : \mathbf{nn}$, potential $\Delta f = (\pi/E)\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} : \boldsymbol{\alpha}$ thus identifying a symmetric second rank crack density tensor $\boldsymbol{\alpha}$ given by Eq. (2) as the proper parameter that replaces, in this case, fourth rank density parameter (10).

## 2. Cavity compliance tensor for an elliptical hole arbitrarily oriented in an anisotropic matrix

### *2.1. Relevant representations of 2D anisotropic elasticity*

Following Lekhnitski (1963), we express stresses and displacements in a 2D anisotropic elastic solid in terms of two complex stress functions $\phi(z_1)$ and $\psi(z_2)$, where $z_1 = x + \mu_1 y$, $z_2 = x + \mu_2 y$. Complex parameters $\mu_1, \mu_2$ and their conjugates $\bar{\mu}_1, \bar{\mu}_2$ are roots of the characteristic equation

$$S_{1111}\mu^4 - 2S_{1112}\mu^3 + (2S_{1122} + S_{1212})\mu^2 - 2S_{2212}\mu + S_{2222} = 0 \tag{13}$$

where $S_{ijkl}$ are elastic compliances in coordinate system $x, y$. Positive definiteness of the strain energy implies that $\mu_1, \mu_2$ cannot be real. We denote $\mu_k = \alpha_k + i\beta_k$ where $\alpha_k, \beta_k$ are real constants and $\beta_k > 0$.

The stresses and displacements are expressed in terms of $\phi(z_1)$ and $\psi(z_2)$ as follows:

$$\sigma_{xx} = 2\mathrm{Re}\left[\mu_1^2\phi'(z_1) + \mu_2^2\psi'(z_2)\right]$$

$$\sigma_{yy} = 2\mathrm{Re}\left[\phi'(z_1) + \psi'(z_2)\right]$$

$$\sigma_{xy} = -2\mathrm{Re}\left[\mu_1\phi'(z_1) + \mu_2\psi'(z_2)\right]$$

$$u_1(x, y) = 2\mathrm{Re}\left[p_1\phi(z_1) + p_2\psi(z_2)\right]$$

$$u_2(x, y) = 2\mathrm{Re}\left[q_1\phi(z_1) + q_2\psi(z_2)\right] \tag{14}$$

where $p_k = S_{1111}\mu_k^2 - S_{1112}\mu_k + S_{1122}$, $q_k = \mu_k^{-1}(S_{1122}\mu_k^2 - S_{2212} + S_{2222})$.

In the case of *orthotropy* (coordinate axes parallel to the orthotropy axes), Eq. (13) becomes biquadratic:

$$S_{1111}\mu^4 + (2S_{1122} + S_{1212})\mu^2 + S_{2222} = 0 \tag{15}$$

and has purely imaginary roots $\mu_{1,2}$, $\bar{\mu}_{1,2} = -\mu_{1,2}$:

$$\mu_{1,2} = \frac{i}{\sqrt{2S_{1111}}}\left\{(2S_{1122} + S_{1212}) \pm \left[(2S_{1122} + S_{1212})^2 - 4S_{1111}S_{2222}\right]^{1/2}\right\}^{1/2} \tag{16}$$

They can be expressed in terms of "engineering constants" frequently used for the orthotropic materials (Young's moduli $E_1$, $E_2$, shear modulus $G_{12}$ and Poisson's ratio $\nu_{12}$, where $x_1x_2$ are the principal axes of orthotropy) by finding $\mu_1\mu_2$ and $\mu_1 + \mu_2$ from Viete's theorem

$$\mu_{1,2} = i\beta_{1,2} = \frac{i}{2}\left[\left(\frac{E_1}{G_{12}} - 2\nu_{12} + 2\sqrt{\frac{E_1}{E_2}}\right)^{1/2} \pm \left(\frac{E_1}{G_{12}} - 2\nu_{12} - 2\sqrt{\frac{E_1}{E_2}}\right)^{1/2}\right] \tag{17}$$

where $E_i$, $G_{ij}$, and $\nu_{ij}$ are Young's moduli, shear moduli and Poisson's ratios of the matrix in the case of plane stress. In plane strain, $G_{ij}$ retains its meaning, whereas $E_i$ and $\nu_{ij}$ are to be understood as $E_1 = E_1/(1 - \nu_{13}\nu_{31})$, $E_2 = E_2/(1 - \nu_{23}\nu_{32})$ and $\nu_{12} = (\nu_{12} + \nu_{13}\nu_{32})/(1 - \nu_{13}\nu_{31})$, $\nu_{21} = (\nu_{21} + \nu_{23}\nu_{31})/(1 - \nu_{23}\nu_{32})$.

If roots $\hat{\mu}_k$ correspond to coordinate system $\hat{x}_1$, $\hat{x}_2$, then roots $\mu_k$ corresponding to the system $x_1$, $x_2$ rotated by angle $\varphi$ counterclockwise with respect to $\hat{x}_1$, $\hat{x}_2$, are related to $\hat{\mu}_k$ by the following simple transformation (Lekhnitski, 1963):

$$\mu_k = \frac{\hat{\mu}_k \cos\varphi - \sin\varphi}{\cos\varphi + \hat{\mu}_k \sin\varphi} \tag{18}$$

This leads to considerable simplifications in the case of the orthotropic matrix since $\hat{\mu}_k$ corresponding to the principal axes of orthotropy are given by simple expressions (17).

Elastic potential in stresses $f_0(\sigma_{ij}) = (1/2)S^0_{ijkl}\sigma_{ij}\sigma_{kl}$ in the case of the orthotropic 2D matrix, expressed in terms of the "engineering constants", has the form

$$f_0(\sigma_{ij}) = \frac{1}{2E^0_1}\sigma^2_{11} + \frac{1}{2E^0_2}\sigma^2_{22} - \frac{\nu^0_{12}}{E^0_1}\sigma_{11}\sigma_{22} + \frac{1}{2G^0_{12}}\sigma^2_{12} \tag{19}$$

Hereafter, superscript "0" at elastic constants denotes constants of the matrix material and constants without superscripts — the effective elastic moduli of a matrix with holes.

### 2.2. Elliptical hole arbitrarily oriented in an anisotropic matrix

For an elliptical hole in a matrix of *arbitrary anisotropy*, components of the hole compliance tensor **H**, in coordinate axes $x_1$, $x_2$ oriented along the ellipses axes $2a$ and $2b$ (Fig. 2), are as follows (see Appendix A for derivation):

$$H_{1111} = \frac{\pi b}{A}S^0_{1111}\left[a + b(\beta_1 + \beta_2)\right]$$

$$H_{1112} = \frac{\pi b}{2A}S^0_{1111}\left[a(\alpha_1 + \alpha_2) + b(\alpha_1\beta_2 + \alpha_2\beta_1)\right]$$

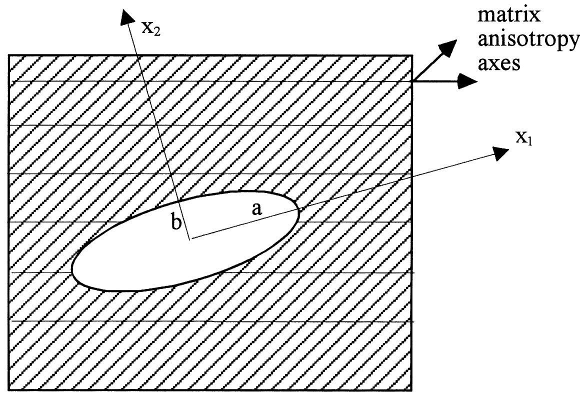


Fig. 2. Elliptical hole arbitrarily oriented with respect to the anisotropy axes of the matrix.

$$H_{1122} = \frac{\pi ab}{A} S^0_{1111}(\alpha_1\alpha_2 - \beta_1\beta_2)$$

$$H_{1212} = \frac{\pi}{4A} S^0_{1111}\left\{a^2(\beta_1+\beta_2) + ab\left[(\alpha_1+\alpha_2)^2 + (\beta_1+\beta_2)^2\right] + b^2\left[\beta_1\left(\alpha_2^2+\beta_2^2\right) + \beta_2\left(\alpha_1^2+\beta_1^2\right)\right]\right\}$$

$$H_{1222} = \frac{\pi a}{2A} S^0_{1111}\left\{a(\alpha_1\beta_2+\alpha_2\beta_1) + b\left[\alpha_1\left(\alpha_2^2+\beta_2^2\right) + \alpha_2\left(\alpha_1^2+\beta_1^2\right)\right]\right\}$$

$$H_{2222} = \frac{\pi a}{A} S^0_{2222}\left[a\left(\frac{\beta_1}{\alpha_1^2+\beta_1^2} + \frac{\beta_2}{\alpha_2^2+\beta_2^2}\right) + b\right] \tag{20}$$

where $\alpha_k$, $\beta_k$ are the real and imaginary parts of roots $\mu_k$ of characteristic equation (13).

In the important case of the *orthotropic* matrix, these formulas can be transformed to explicitly reflect the orientation of the hole with respect to the matrix orthotropy axes. Namely, introducing unit vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ along $2a$ and $2b$ axes of the ellipse and angle $\varphi$ between $\boldsymbol{t}$ and $x_1$-direction of the matrix orthotropy (Fig. 3), we obtain, after some algebra,

$$H_{tttt} = \frac{\pi b}{A}\left\{\frac{a}{E_t^0} + b\left[C(1 - D\cos 2\varphi)\right]\right\}$$

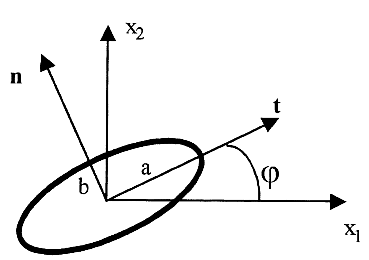


Fig. 3. Elliptical hole in the orthotropic matrix ($x_1$, $x_2$ are the orthotropy axes).

$$H_{tttn} = \frac{\pi b}{4A}\left\{a\left[\frac{1}{E_2^0} - \frac{1}{E_1^0} - F\cos 2\varphi\right] + 2bCD\right\}\sin 2\varphi$$

$$H_{ttnn} = \frac{\pi ab}{4A}\left(F\sin^2 2\varphi - \frac{4}{\sqrt{E_1^0 E_2^0}}\right)$$

$$H_{tntn} = \frac{\pi a^2}{4A}C(1 - D\cos 2\varphi) + \frac{\pi b^2}{4A}C(1 + D\cos 2\varphi) + \frac{\pi ab}{4A}\left[\left(\frac{1}{\sqrt{E_1^0}} + \frac{1}{\sqrt{E_2^0}}\right)^2 - F\cos^2 2\varphi\right]$$

$$H_{tnnn} = \frac{\pi a}{4A}\left\{2aCD + b\left[\frac{1}{E_2^0} - \frac{1}{E_1^0} + F\cos 2\varphi\right]\right\}\sin 2\varphi$$

$$H_{nnnn} = \frac{\pi a}{A}\left\{a\big[C(1 + D\cos 2\varphi)\big] + \frac{b}{E_n^0}\right\} \tag{21}$$

where constants $C$, $D$ and $F$ are expressed in terms of the "engineering constants" of the matrix, $E_1^0$, $E_2^0$, $G_{12}^0$, $\nu_{12}^0$ as follows:

$$C = \frac{1}{2}\frac{\sqrt{E_1^0} + \sqrt{E_2^0}}{\sqrt{E_1^0 E_2^0}}\sqrt{\frac{1}{G_{12}^0} - \frac{2\nu_{12}^0}{E_1^0} + \frac{2}{\sqrt{E_1^0 E_2^0}}}$$

$$D = \frac{\sqrt{E_1^0} - \sqrt{E_2^0}}{\sqrt{E_1^0} + \sqrt{E_2^0}}$$

$$F = \frac{1 + \nu_{12}^0}{E_1^0} + \frac{1 + \nu_{21}^0}{E_2^0} - \frac{1}{G_{12}^0} \tag{22}$$

and where $E_t^0$ and $E_n^0$ are Young's moduli of the matrix in ***t***- and ***n***-directions given by

$$\frac{1}{E_t^0} = \frac{\cos^4\varphi}{E_1^0} + \left(\frac{1}{G_{12}^0} - \frac{2\nu_{12}^0}{E_1^0}\right)\sin^2\varphi\cos^2\varphi + \frac{\sin^4\varphi}{E_2^0},$$

and

$$\frac{1}{E_n^0} = \frac{\sin^4\varphi}{E_1^0} + \left(\frac{1}{G_{12}^0} - \frac{2\nu_{12}^0}{E_1^0}\right)\sin^2\varphi\cos^2\varphi + \frac{\cos^4\varphi}{E_2^0}.$$

**Remark.** Although the set of 2D orthotropic moduli consists, generally, of four independent constants, $H_{ijkl}$ — components (21) can, in principle, be expressed in terms of only three combinations of the matrix moduli, $E_1^0$, $E_1^0$, $1/G_{12}^0 - 2\nu_{12}^0/E_1^0$. Form (21), that contains larger number of matrix constants, is given here, because it is shorter. It also has the advantage that constants C and D are measures of the extent of the matrix anisotropy (in the case of cubic symmetry, $D = 0$; in the case of isotropy, $C = 0$, $D = 0$, $F = 0$).

We will also need components of **H**-tensor in the principal axes of matrix orthotropy $x_1 x_2$. They are as follows:

$$H_{1111} = \frac{\pi L}{A\sqrt{E_1^0}}\left[(b^2 - a^2)\cos^2\varphi + a^2 + \frac{ab}{L\sqrt{E_1^0}}\right]$$

$$H_{1112} = \frac{\pi(b^2 - a^2)L}{2A\sqrt{E_1^0}}\sin\varphi\cos\varphi$$

$$H_{1122} = -\frac{\pi ab}{A\sqrt{E_1^0 E_2^0}}$$

$$H_{1212} = \frac{\pi L}{4A\sqrt{E_1^0 E_2^0}}\left[(a^2 - b^2)\left(\sqrt{E_2^0} - \sqrt{E_1^0}\right)\cos^2\varphi + a^2\sqrt{E_1^0} + abL\sqrt{E_1^0 E_2^0} + b^2\sqrt{E_2^0}\right]$$

$$H_{1222} = \frac{\pi(b^2 - a^2)L}{2A\sqrt{E_2^0}}\sin\varphi\cos\varphi$$

$$H_{2222} = \frac{\pi L}{A\sqrt{E_2^0}}\left[(a^2 - b^2)\cos^2\varphi + b^2 + \frac{ab}{L\sqrt{E_2^0}}\right] \tag{23}$$

and are expressed in terms of three matrix constants: $E_1^0$, $E_2^0$ and combination

$$L = \sqrt{\frac{1}{G_{12}^0} - \frac{2\nu_{12}^0}{E_1^0} + \frac{2}{\sqrt{E_1^0 E_2^0}}}.$$

We now consider two limiting cases: circular hole and a crack. Besides being of interest of their own, these two cases are important because compliance of an elliptical hole can be represented as a sum of compliances of a circle and of two cracks (Section 2.5).

## *2.3. The case of a crack arbitrarily oriented in the orthotropic matrix*

In the case of a crack, $b = 0$, $a = l$ and **H**-tensor has form (9). If the crack is arbitrarily oriented in the orthotropic matrix, the dependence of **H** on the crack orientation (angle $\varphi$ between the crack line and the $x_1$ principal axis of orthotropy) is remarkably simple, and only *two combinations*, $C$ and $D$, of the (four) matrix moduli are present in **H**:

$$H_{tntn} = \frac{\pi l^2}{A} C(1 - D\cos 2\varphi)$$

$$H_{tnnn} = \frac{\pi l^2}{2A} CD \sin 2\varphi$$

$$H_{nnnn} = \frac{\pi l^2}{A} C(1 + D\cos 2\varphi)$$

$$\text{Other } H_{ijkl} = 0 \tag{24}$$

This recovers results of Mauge and Kachanov (1994).

Expressions (24) imply that *crack compliance tensor* **B** (defined by (7)) is *constant* (independent of crack orientation $\varphi$):

$$\mathbf{B} = \frac{\pi C}{2}(1 + D)\mathbf{e}_1\mathbf{e}_1 + \frac{\pi C}{2}(1 - D)\mathbf{e}_2\mathbf{e}_2 \tag{25}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ are unit vectors along the matrix orthotropy axes. This means that **H**-tensor reflects the crack orientation in a very simple way: through dyadic product of **B** with **n** *only*, see (9). As seen in Section 3, constancy of **B** has important implications for the choice of the proper crack density parameters.

Note that, since **B** is not proportional to unit tensor **I**, normal and shear modes are coupled: normal/shear traction on a crack produces shear/normal CODs for an arbitrarily oriented crack (except for a crack parallel to one of the matrix orthotropy axes).

As seen from (22), of the four matrix constants, Young's moduli $E_1^0$, $E_2^0$ play a special role: the case $E_1^0 = E_2^0$, with no restrictions on shear modulus $G_{12}^0$ (cubic symmetry) is similar, to within a constant multiplier, to the case of isotropy. In this case, crack compliance tensor **B** is proportional to unit tensor **I**:

$$\mathbf{B} = \pi\mathbf{I}\begin{cases} \dfrac{1}{E^0} & \text{isotropic matrix} \\ \dfrac{1}{2E_1^0}\sqrt{\dfrac{E_1^0}{G_{12}^0} + 2 - 2\nu_{12}^0} & \text{matrix of cubic symmetry} \end{cases} \tag{26}$$

**Remark.** In the case of moderate matrix anisotropy in Young's moduli ($E_1^0$ and $E_2^0$ differ by 40–50% or less with no restrictions on shear modulus $G_{12}^0$), D is one order of magnitude smaller than unity, and, with this accuracy, $\mathbf{B} \approx \mathbf{I}$, implying that the abovementioned coupling of the normal and shear modes can be neglected. This case is thus approximately equivalent to the one of matrix isotropy.

### 2.4. The case of a circular hole in the orthotropic matrix

In the case of a circular hole (of radius $a$), components of **H**-tensor in the matrix orthotropy axes $x_1x_2$ take the form

$$H_{1111} = \frac{\pi a^2}{AE_1^0}\left(1 + L\sqrt{E_1^0}\right)$$

$$H_{1122} = -\frac{\pi a^2}{A\sqrt{E_1^0 E_2^0}}$$

$$H_{1212} = \frac{\pi a^2}{4A}L\left(\frac{1}{\sqrt{E_1^0}} + \frac{1}{\sqrt{E_2^0}} + L\right)$$

$$H_{2222} = \frac{\pi a^2}{AE_2^0}\left(1 + L\sqrt{E_2^0}\right) \tag{27}$$

**Remark 1.** Although the circular hole is "geometrically isotropic", its influence on the elastic compliance is anisotropic (potential change $\Delta f$, given by (6), is anisotropic). This anisotropy is opposite to the one of the matrix: as seen from comparison of $H_{1111}$ and $H_{2222}$, the reduction of stiffness due to the hole is greater in the stiffer direction of the matrix. This implies that circular holes reduce the extent of the matrix anisotropy. Physically, it is explained by the fact that "loss of cross-section" in the direction normal to the stiffer direction of the matrix produces a larger contribution to the overall compliance than an equal loss in the "softer" direction.

**Remark 2.** In the case of cubic symmetry of the matrix ($E_1^0 = E_2^0$, but $G_{12}^0$ remains an independent constant), **H**-tensor of a circular hole does not reduce to the one for the isotropic matrix. This sensitivity to the difference between cubic symmetry and isotropy constitutes an unexpected contrast with the case of a crack, for which **H**-tensor has the same form (to within a multiplier) for the matrix of cubic symmetry and for the isotropic matrix (see (26)).

In the case of the *isotropic* matrix, $H_{1111} = H_{2222} = 3\pi a^2/(AE^0)$, $H_{1122} = \pi a^2/(AE^0)$, $H_{1212} = 2\pi a^2/(AE^0)$.

### 2.5. Compliance of an elliptical hole is a sum of compliances of a circular hole and two cracks

Based on the results of the preceding subsections, compliance tensor of an elliptical hole with axes $2a$, $2b$ can be represented as a sum of compliance tensors of a circular hole of radius $\sqrt{ab}$ and of two mutually perpendicular cracks: crack 1 of length $2\sqrt{a(a-b)}$, parallel to $2a$-axis and crack 2 of length $2\sqrt{b(a-b)}$, parallel to $2b$-axis; compliance of crack 2 being taken with the negative sign (Fig. 4):

$$\mathbf{H} = \mathbf{H}^{\text{circle}} + \mathbf{H}^{\text{crack 1}} - \mathbf{H}^{\text{crack 2}} \tag{28}$$

or, in components,

$$H_{1111} = \frac{\pi L}{A\sqrt{E_1^0}}\left[ab\left(1+\frac{1}{L\sqrt{E_1^0}}\right)+a(a-b)\sin^2\varphi - b(a-b)\cos^2\varphi\right]$$

$$H_{1112} = -\frac{\pi L}{2A\sqrt{E_1^0}}[0+a(a-b)+b(a-b)]\sin\varphi\cos\varphi$$

$$H_{1122} = -\frac{\pi}{A\sqrt{E_1^0E_2^0}}[ab+0-0]$$

$$H_{1212} = \frac{\pi L}{4A\sqrt{E_1^0E_2^0}}\Bigg[ab\left(L\sqrt{E_1^0E_2^0}+\sqrt{E_1^0}+\sqrt{E_2^0}\right)+a(a-b)\left(\sqrt{E_1^0}\sin^2\varphi+\sqrt{E_2^0}\cos^2\varphi\right)$$

$$-\,b(a-b)\left(\sqrt{E_1^0}\cos^2\varphi+\sqrt{E_2^0}\sin^2\varphi\right)\Bigg]$$

$$H_{1222} = -\frac{\pi L}{2A\sqrt{E_2^0}}[0+a(a-b)+b(a-b)]\sin\varphi\cos\varphi$$

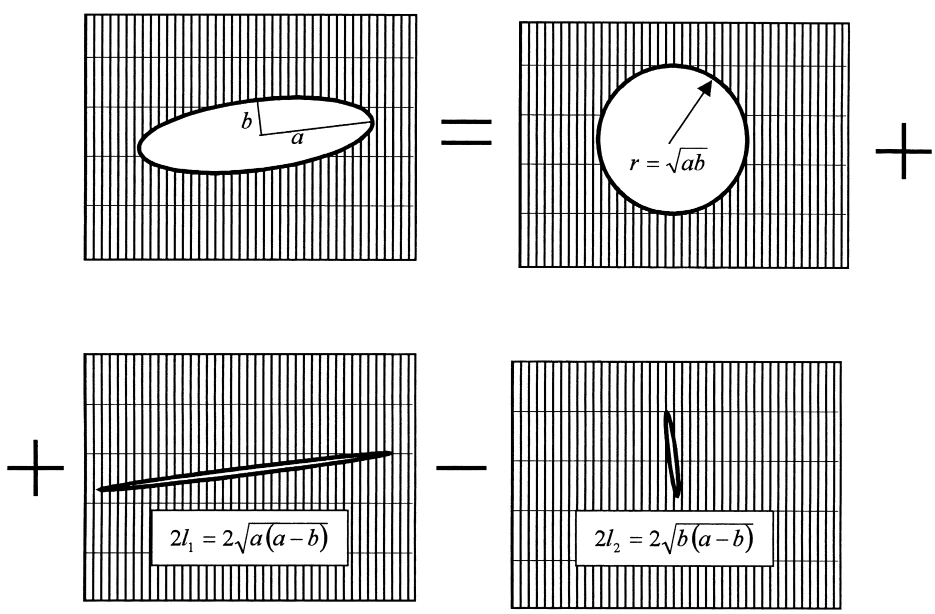


Fig. 4. Representation of ellipse's compliance tensor **H** as a sum of **H** of a circular hole (of radius $\sqrt{ab}$) plus **H** of a crack parallel to $2a$ axis (of length $2\sqrt{a(a-b)}$) minus **H** of a crack parallel to $2b$ axis (of length $2\sqrt{b(a-b)}$).

$$H_{2222} = \frac{\pi L}{A\sqrt{E_2^0}}\left[ab\left(1 + \frac{1}{L\sqrt{E_2^0}}\right) + a(a-b)\cos^2\varphi - b(a-b)\sin^2\varphi\right] \tag{29}$$

The three terms in each of the brackets represent the contribution of the circular hole and of the mentioned two cracks, correspondingly. In the cases when some of these contributions vanish, the corresponding terms are entered as zeros (for example, two zeros in the expression for $H_{1122}$ reflect the fact that Poisson's ratio effect is not affected by cracks).

In the case of the isotropic matrix, this representation implies that the anisotropy due to non-randomly oriented elliptical holes is the same as the anisotropy due to a certain set of cracks. Since a solid with non-interacting cracks is orthotropic for any orientational distribution of cracks (Kachanov, 1980), this implies orthotropy for a solid with ellipses, recovering the results of Kachanov et al. (1994).

Thus, *any set of non-interacting elliptical holes can be represented as a mixture of non-interacting circular holes and cracks* of appropriate sizes and orientations.

**Remark.** Sizes of the circle and of the cracks do not depend on the matrix constants. In particular, representation (29) holds for the isotropic matrix.

## 3. One family of parallel ellipses of an arbitrary orientation

This case is of a fundamental importance, since results for any orientational distribution of non-interacting ellipses can be obtained by integration over orientations (with appropriate distribution density) of the results of this section.

For one family of parallel holes inclined at an angle $\varphi$ with respect to $x_1$ axis of the matrix orthotropy, tensor $\mathbf{H}^* = \sum \mathbf{H}^{(k)}$ entering expression (10) for potential $\Delta f$, can be expressed in terms of porosity $p = (1/A)\pi \sum (ab)^{(k)}$ (area fraction of holes) and symmetric second rank *hole density tensor*

$$\boldsymbol{\beta} = \frac{\pi}{A}\sum_k \left(a^2\mathbf{nn} + b^2\mathbf{tt}\right)^{(k)}.$$

where $2a$ and $2b$ are ellipses' axes along directions of unit vectors $\mathbf{t}$ and $\mathbf{n}$, correspondingly.

$$H^*_{1111} = \frac{1}{E_1^0}\left[p + L\sqrt{E_1^0}\left(\beta_{nn}\sin^2\varphi + \beta_{tt}\cos^2\varphi\right)\right]$$

$$H^*_{1112} = \frac{L}{2\sqrt{E_1^0}}\left(\beta_{tt} - \beta_{nn}\right)\sin\varphi\cos\varphi$$

$$H^*_{1212} = \frac{L}{4\sqrt{E_1^0E_2^0}}\left[pL\sqrt{E_1^0E_2^0} + \sqrt{E_1^0}\left(\beta_{nn}\sin^2\varphi + \beta_{tt}\cos^2\varphi\right) + \sqrt{E_2^0}\left(\beta_{nn}\cos^2\varphi + \beta_{tt}\sin^2\varphi\right)\right]$$

$$H^*_{1122} = -\frac{p}{\sqrt{E_1^0E_2^0}}$$

$$H^*_{1222} = \frac{L}{2\sqrt{E^0_2}}(\beta_{tt} - \beta_{nn}) \sin\varphi \cos\varphi$$

$$H^*_{2222} = \frac{1}{E^0_2}\left[p + L\sqrt{E^0_2}(\beta_{nn}\cos^2\varphi + \beta_{tt}\sin^2\varphi)\right] \tag{30}$$

where $\beta_{nn} = (1/A)\pi\sum a^{(k)2}$ and $\beta_{tt} = (1/A)\pi\sum b^{(k)2}$ are components of $\boldsymbol{\beta}$.

The effective elastic constants, generally, are non-orthotropic (except for the case when the holes are aligned with the matrix orthotropy axes). They immediately follow from Eqs. (10) and (11), with $\mathbf{H}^*$ given by Eq. (30). A partial set of the effective elastic moduli, given in terms of the "engineering constants", is

$$\frac{E_1}{E^0_1} = \left[1 + p + L\sqrt{E^0_1}(\beta_{nn}\sin^2\varphi + \beta_{tt}\cos^2\varphi)\right]^{-1}$$

$$\frac{E_2}{E^0_2} = \left[1 + p + L\sqrt{E^0_2}(\beta_{nn}\cos^2\varphi + \beta_{tt}\sin^2\varphi)\right]^{-1}$$

$$\frac{G_{12}}{G^0_{12}} = \left[1 + \frac{pL^2G^0_{12}}{2} + \frac{LG^0_{12}}{2\sqrt{E^0_1}}(\beta_{nn}\cos^2\varphi + \beta_{tt}\sin^2\varphi) + \frac{LG^0_{12}}{2\sqrt{E^0_2}}(\beta_{nn}\sin^2\varphi + \beta_{tt}\cos^2\varphi)\right]^{-1}$$

$$\frac{\nu_{12}}{E_1} = \frac{\nu^0_{12}}{E^0_1} + \frac{p}{\sqrt{E^0_1 E^0_2}} \tag{31}$$

### 3.1. Circular holes in the orthotropic matrix

In the case of circular holes of radii $a^{(k)}$, porosity $p = (1/A)\pi\sum a^{(k)2}$ and hole density tensor is proportional to unit tensor $\mathbf{I}: \boldsymbol{\beta} = p\mathbf{I}$, so that the potential change due to holes takes the form:

$$\Delta f = \frac{1}{2}p\boldsymbol{\sigma} : \left[\frac{1 + L\sqrt{E^0_1}}{E^0_1}\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1 - \frac{1}{\sqrt{E^0_1E^0_2}}(\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1) + \frac{L}{4}\left(\frac{1}{\sqrt{E^0_1}} + L + \frac{1}{\sqrt{E^0_2}}\right)(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1)(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) + \frac{1 + L\sqrt{E^0_2}}{E^0_2}\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2\mathbf{e}_2\right] : \boldsymbol{\sigma}$$

$$= \frac{1}{2}p\left[\frac{1+L\sqrt{E_1^0}}{E_1^0}\sigma_{11}^2 - \frac{2}{\sqrt{E_1^0E_2^0}}\sigma_{11}\sigma_{22} + \left(\frac{1}{\sqrt{E_1^0}} + L + \frac{1}{\sqrt{E_2^0}}\right)L\sigma_{12}^2 + \frac{1+L\sqrt{E_2^0}}{E_2^0}\sigma_{22}^2\right] \tag{32}$$

The effective moduli, expressed in terms of $p$, immediately follow:

$$E_1 = E_1^0\left[1+p\left(1+L\sqrt{E_1^0}\right)\right]^{-1}$$

$$E_2 = E_2^0\left[1+p\left(1+L\sqrt{E_2^0}\right)\right]^{-1}$$

$$G_{12} = G_{12}^0\left[1+\frac{pLG_{12}^0}{2}\left(\frac{1}{\sqrt{E_1^0}} + L + \frac{1}{\sqrt{E_2^0}}\right)\right]^{-1}$$

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{12}^0}{E_1^0} + \frac{p}{\sqrt{E_1^0E_2^0}} \tag{33}$$

These results are illustrated by the dependence of $E_1/E_2$ on porosity $p$ in Fig. 5 (upper line).

### *3.2. Cracks arbitrarily oriented in the orthotropic matrix*

As discussed above, fourth rank tensor $(1/A)\sum(l^2\mathbf{nBn})^{(k)}$ in Eq. (12) is the proper density parameter. Effective properties are expressed in terms of this parameter in a unified way for all orientational distributions of cracks. In certain simplest cases of the orientational distributions (notably, random orientations and one family of parallel cracks), these expressions can be reduced to formulas in terms of the conventional crack density $\rho$ given by Eq. (1).

Since **H**-tensor of a crack has the form (9), the proper crack density parameter is the following fourth

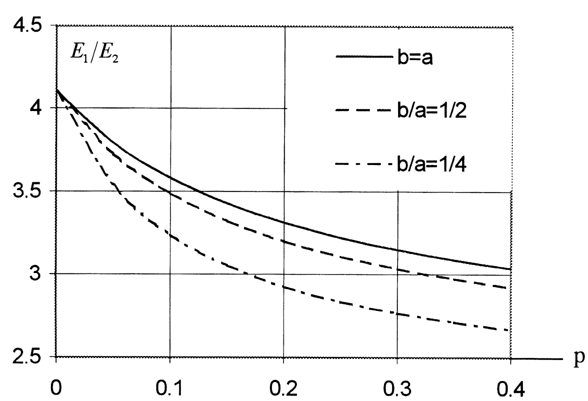


Fig. 5. Ratio $E_1/E_2$ for the orthotropic matrix $E_1^0/E_2^0 = 4.1$, $E_1^0/G_{12}^0 = 10$, $\nu_{12}^0 = 0.277$, $\nu_{21}^0 = 0.068$) with randomly oriented elliptical holes, as function of porosity $p$, for holes of aspect ratios $b/a$ equal to 1 (circles), 1/2 and 1/4.

rank tensor:

$$\mathbf{H}^* = \frac{2}{A}\sum (l^2\mathbf{nBn})^{(k)} \quad \text{(appropriately symmetrized)} \tag{34}$$

or, in components,

$$H^*_{ijkl} = \frac{1}{3}\left(B_{ij}\alpha_{kl} + B_{kl}\alpha_{ij} + B_{ik}\alpha_{jl} + B_{il}\alpha_{jk} + B_{jk}\alpha_{il} + B_{jl}\alpha_{ik}\right) \tag{35}$$

where $\boldsymbol{\alpha}$ is the second rank crack density tensor (2).

Potential $\Delta f$ takes the form (with the account of (25)):

$$\Delta f = \frac{1}{6}\left[2(\boldsymbol{\sigma}:\mathbf{B})(\boldsymbol{\sigma}:\boldsymbol{\alpha}) + (\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}):\mathbf{B} + (\boldsymbol{\sigma}\cdot\mathbf{B}\cdot\boldsymbol{\sigma}):\boldsymbol{\alpha} + (\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}):(\mathbf{B}\cdot\boldsymbol{\sigma}) + (\boldsymbol{\sigma}\cdot\mathbf{B}):(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})\right]$$

$$= \frac{\pi C}{12}\left\{6(1+D)\alpha_{11}\sigma_{11}^2 + 6(1-D)\alpha_{22}\sigma_{22}^2 + 4(\alpha_{11}+\alpha_{22})\sigma_{12}^2 + \left[(1+D)\alpha_{22} + (1-D)\alpha_{11}\right]\sigma_{11}\sigma_{22}\right\} \tag{36}$$

Potential (36) yields effective moduli for any orientational distribution of cracks in a unified way.

In the case of *one family* of arbitrarily oriented *parallel* cracks, the effective properties are non-orthotropic. A partial set of the effective moduli given in terms of the ‘‘engineering constants’’ is

$$\frac{E_1}{E_1^0} = \left[1 + \pi\rho C(1-D)E_1^0 \sin^2\varphi\right]^{-1}$$

$$\frac{E_2}{E_2^0} = \left[1 + \pi\rho C(1+D)E_2^0 \cos^2\varphi\right]^{-1}$$

$$\frac{G_{12}}{G_{12}^0} = \left[1 + \left(\frac{\pi}{2}\right)\rho C(1 - D\cos 2\varphi)G_{12}^0\right]^{-1}$$

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{12}^0}{E_1^0} \tag{37}$$

In particular, in the case when cracks are parallel to one of the orthotropy axes of the matrix $x_1$-axis),

$$E_1 = E_1^0$$

$$E_2 = E_2^0\left(1 + \pi L\sqrt{E_2^0}\rho\right)^{-1}$$

$$G_{12} = G_{12}^0\left(1 + \frac{\pi L G_{12}^0}{2\sqrt{E_1^0}}\rho\right)^{-1}$$

$$\nu_{12} = \nu_{12}^0 \tag{38}$$

recovering results of Gottesman et al. (1980).

In the case of *randomly oriented cracks*,

$$\frac{E_1}{E_1^0} = \left[1 + \left(\frac{\pi L}{2}\right)\sqrt{E_1^0}\rho\right]^{-1}$$

$$\frac{E_1}{E_1^0} = \left[1 + \left(\frac{\pi L}{2}\right)\sqrt{E_2^0}\rho\right]^{-1}$$

$$\frac{G_{12}}{G_{12}^0} = \left[1 + \frac{\pi L G_{12}^0}{4\sqrt{E_1^0 E_2^0}}\left(\sqrt{E_2^0} + \sqrt{E_2^0}\right)\rho\right]^{-1}$$

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{12}^0}{E_1^0} \tag{39}$$

Note that in two special cases considered above — parallel and randomly oriented cracks — the effective moduli can be expressed in terms of conventional scalar density $\rho$. In general, however, parameter $\rho$ is not sufficient (as in the case of two families of parallel cracks at an angle to each other).

### 3.3. *The case of elliptical holes in the isotropic matrix*

In this case, $E_1^0 = E_2^0 = E^0$, $G_{12}^0 = 0.5E^0/(1 + \nu^0)$, $L = 2/\sqrt{E^0}$ and

$$E_1 = E^0\left(1 + p + 2\beta_{11}\right)^{-1}$$

$$E_2 = E^0\left(1 + p + 2\beta_{22}\right)^{-1}$$

$$G_{12} = G^0\left[1 + \frac{1}{2(1 + \nu^0)}\left(\beta_{11} + 2p + \beta_{22}\right)\right]^{-1}$$

$$\nu_{12} = \left(\nu^0 + p\right)\left(1 + p + 2\beta_{11}\right)^{-1} \tag{40}$$

recovering results of Tsukrov and Kachanov (1993) and Kachanov et al. (1994).

## 4. Randomly oriented ellipses: gradual disappearance of anisotropy as the hole density increases

In the case of randomly oriented ellipses, the components of hole density tensor $\mathbf{H}^* = \sum \mathbf{H}^{(k)}$, calculated by replacing the summation over ellipses by integration over orientations (in the assumption that the orientational distribution of holes is uncorrelated with their sizes and aspect ratio distributions), are given in terms of porosity $p$ and eccentricity parameter

$$q = \frac{\pi}{A}\sum (a-b)^{2(k)} \tag{41}$$

as follows

$$H^*_{1111} = \frac{1 + L\sqrt{E^0_1}}{E^0_1}p + \frac{L}{2\sqrt{E^0_1}}q$$

$$H^*_{1122} = -\frac{1}{\sqrt{E^0_1 E^0_2}}p$$

$$H^*_{1212} = \frac{L}{8\sqrt{E^0_1 E^0_2}}\left[2\left(\sqrt{E^0_2} + L\sqrt{E^0_1 E^0_2} + \sqrt{E^0_1}\right)p + \left(\sqrt{E^0_2} + \sqrt{E^0_1}\right)q\right]$$

$$H^*_{2222} = \frac{1 + L\sqrt{E^0_2}}{E^0_2}p + \frac{L}{2\sqrt{E^0_2}}q \tag{42}$$

Note that, in spite of random orientations, tensor $\mathbf{H}^* = \sum \mathbf{H}^{(k)}$ is not isotropic. Physically, this means that randomly oriented ellipses produce a higher impact on the compliance in the "stiffer" direction of the matrix and, thus, reduce the overall anisotropy (as seen from formulas (43) below).

The effective elastic moduli are orthotropic and given by

$$\frac{E_1}{E^0_1} = \left[1 + \left(1 + L\sqrt{E^0_1}\right)p + \left(\frac{L}{2}\right)\sqrt{E^0_1}q\right]^{-1}$$

$$\frac{E_1}{E^0_1} = \left[1 + \left(1 + L\sqrt{E^0_2}\right)p + \left(\frac{L}{2}\right)\sqrt{E^0_2}q\right]^{-1}$$

$$\frac{G_{12}}{G^0_{12}} = \left\{1 + \frac{LG^0_{12}}{4\sqrt{E^0_1 E^0_2}}\left[\left(\sqrt{E^0_2} + L\sqrt{E^0_1 E^0_2} + \sqrt{E^0_2}\right)p + \left(\sqrt{E^0_2} + \sqrt{E^0_2}\right)q\right]\right\}^{-1}$$

$$\frac{\nu_{12}}{E_1} = \frac{\nu^0_{12}}{E^0_1} + \frac{p}{\sqrt{E^0_1 E^0_2}} \tag{43}$$

We emphasize that, even in the case of random orientations of holes, the moduli cannot be expressed in terms of porosity $p$ alone — a second hole density parameter $q$ is also needed.

The ratio of Young's moduli $E_1/E_2$ indicating the extent of anisotropy is:

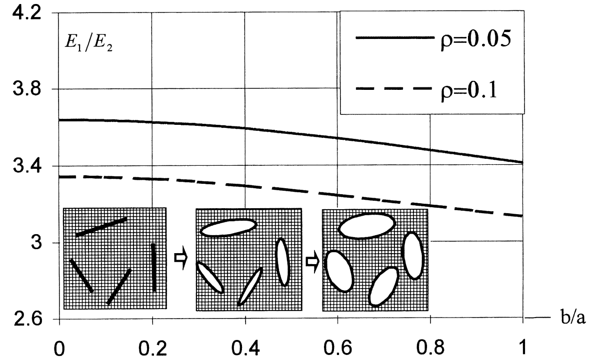


Fig. 6. Ratio $E_1/E_2$ for the orthotropic matrix ($E_1^0/E_2^0 = 4.1$, $E_1^0/G_{12}^0 = 10$, $\nu_{12}^0 = 0.277$, $\nu_{21}^0 = 0.068$) with randomly oriented elliptical holes, as function of ellipses' aspect ratio $b/a$. Each of the two curves corresponds to keeping the density $\rho$ of "cracks" (lines of the major ellipses' axes) constant, while $b/a$ increases.

$$\frac{E_1}{E_2} = \frac{E_1^0}{E_2^0}\,\frac{1+\left(1+L\sqrt{E_2^0}\right)p+(L/2)\sqrt{E_2^0}q}{1+\left(1+L\sqrt{E_1^0}\right)p+(L/2)\sqrt{E_1^0}q} \tag{44}$$

Figs. 5–7 demonstrate dependence of the extent of the overall anisotropy (as measured by ratio $E_1/E_2$) on several geometric parameters. Fig. 5 shows that randomly oriented ellipses reduce the effective anisotropy and that this effect is more pronounced, at the same porosity $p$, for narrower ellipses (smaller $b/a$). This generalizes a similar observation of Mauge and Kachanov (1994) on cracks ($b/a = 0$) in an anisotropic matrix. The dependence of $E_1/E_2$ on the aspect ratio of the ellipses (major axes of the ellipses are kept constant, minor axes increase) is illustrated in Fig. 6. An interesting observation is that the decrease in $E_1/E_2$ does not change much, as holes are transformed from cracks to circles (as long as the major ellipses' axes are kept constant). Fig. 7 shows $E_1/E_2$ as a function of aspect ratio $b/a$ at constant porosity. Note that, in the range of $b/a$ from 0.5 to 1.0, ratio $E_1/E_2$ is almost insensitive to $b/a$.

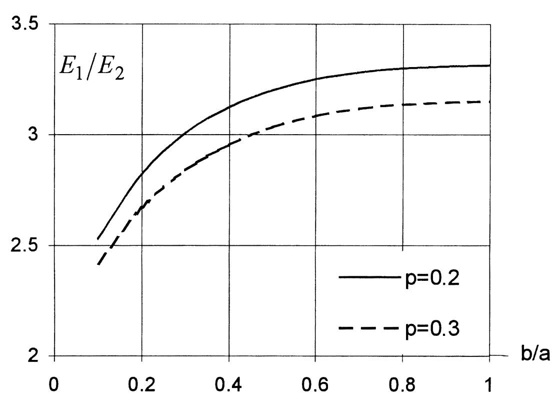


Fig. 7. Ratio $E_1/E_2$ for the orthotropic matrix ($E_1^0/E_2^0 = 4.1$, $E_1^0/G_{12}^0 = 10$, $\nu_{12}^0 = 0.277$, $\nu_{21}^0 = 0.068$) with randomly oriented elliptical holes, as function of ellipses' aspect ratio $b/a$ for porosity of holes $p = (\pi/A)\sum(ab)^{(k)} = 0.2$ (solid line) or 0.3 (dashed line).

## 5. Conclusions

Effective elastic properties of an anisotropic matrix with elliptical holes of an arbitrary orientational distribution are derived in closed form. One of the key problems is the identification of the proper parameters of defect density that correctly reflect the individual defect contributions into the overall compliances. Expressions of the effective moduli in terms of such parameters apply to all orientational distributions of holes. Another advantage is that any mixtures of holes of diverse eccentricities (including cracks) are covered in a unified way.

Several physically interesting effects are identified and discussed. Among them:

- Elongated holes normal to the stiffer direction of the matrix produce a higher impact on the effective compliance than the ones normal to the softer direction.
- Randomly oriented holes reduce the extent of the overall anisotropy.
- At the same porosity, holes of smaller aspect ratios produce a larger contribution into the overall compliances.
- Circular holes are "anisotropic objects", in the sense that their influence on the overall compliance (reflected in potential $\Delta f$) is anisotropic (the highest increase of compliance is in the stiffer direction of the matrix).

## Acknowledgements

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## Appendix A. Derivation of hole compliance tensor *H* for an elliptical hole in a matrix possessing arbitrary elastic anisotropy

We derive here **H**-tensor given by expression (20) of the main text, by utilizing the complex variable formalism developed for anisotropic solids by Lekhnitski (1936) and Savin (1961).

Stress functions $\phi(z_1)$ and $\psi(z_2)$ for a 2D anisotropic solid with an elliptical hole (axes $2a$ and $2b$ are directed along $x_1$ and $x_2$ axes, correspondingly) under a uniform uniaxial loading $P$ inclined at angle $\alpha$ to $x_1$-axis were derived by Lekhnitski (1936) using conformal mapping of $z$-plane onto $\zeta$-plane that transforms the exterior of the elliptical hole into the interior of unit circle $\zeta = \mathrm{e}^{i\theta}$:

$$\phi(z_1) = A_1 z_1 + \phi_0(z_1), \qquad \psi(z_2) = (A_2 + iA_3)z_2 + \psi_0(z_2) \tag{A1}$$

where

$$\phi_0(z_1) = -P\frac{i(a - i\mu_1 b)}{4(\mu_1 - \mu_2)}\frac{b(\mu_2 \sin 2\alpha + 2\cos^2\alpha) + ia(2\mu_2 \sin^2\alpha + \sin 2\alpha)}{z_1 + \sqrt{z_1^2 - (a + \mu_1^2 b^2)}}$$

$$\psi_0(z_2) = P\frac{i(a - i\mu_2 b)}{4(\mu_1 - \mu_2)}\frac{b(\mu_1 \sin 2\alpha + 2\cos^2\alpha) + ia(2\mu_1 \sin^2\alpha + \sin 2\alpha)}{z_2 + \sqrt{z_2^2 - (a + \mu_2^2 b^2)}}$$

are functions of $z_1 = \frac{a+i\mu_1 b}{2}\zeta + \frac{a-i\mu_1 b}{2}\frac{1}{\zeta}$ and $z_2 = \frac{a+i\mu_2 b}{2}\zeta + \frac{a-i\mu_2 b}{2}\frac{1}{\zeta}$, respectively. Constants $A_1$, $A_2$, $A_3$ are given by

$$A_1 = \frac{P}{2}\,\frac{\cos^2\alpha + \left(\alpha_2^2 + \beta_2^2\right)\sin^2\alpha + \alpha_2 \sin 2\alpha}{(\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2}$$

$$A_2 = \frac{P}{2}\,\frac{\left(\alpha_1^2 - 2\alpha_1\alpha_2 - \beta_1^2\right)\sin^2\alpha - \cos^2\alpha - \alpha_2 \sin 2\alpha}{(\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2}$$

$$A_3 = \frac{P}{2\beta_2}\frac{(\alpha_1 - \alpha_2)\cos^2\alpha + [\alpha_2\left(\alpha_1^2 - \beta_1^2\right) - \alpha_1\left(\alpha_2^2 - \beta_2^2\right)]\sin^2\alpha}{(\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2} + \frac{P}{2\beta_2}\frac{\left(\alpha_1^2 - \beta_1^2 - \alpha_2^2 + \beta_2^2\right)\sin\alpha\cos\alpha}{(\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2}.$$

Values of stress functions $\varphi$ and $\phi$ at the hole boundary are as follows (angle $\theta$ characterizes a current point on the boundary):

$$\varphi = \varphi_{rc}\cos\theta + \varphi_{rs}\sin\theta + i(\varphi_{ic}\cos\theta + \varphi_{is}\sin\theta)$$

$$\phi = \phi_{rc}\cos\theta + \phi_{rs}\sin\theta + i(\phi_{ic}\cos\theta + \phi_{is}\sin\theta) \tag{A2}$$

expressed in terms of the following functions of angle $\alpha$:

$$\varphi_{rc} = PM\Big\{2a\left(\alpha_1\alpha_2 - \alpha_2^2 + \beta_1\beta_2 - \beta_2^2\right)\sin^2\alpha + \left[a(\alpha_1 - \alpha_2) + b(\alpha_1\beta_2 - \alpha_2\beta_1)\right]\sin 2\alpha$$
$$+ 2b(\beta_2 - \beta_1)\cos^2\alpha\Big\} + PNa[\left(\alpha_2^2 + \beta_2^2\right)\sin^2\alpha + \alpha_2 \sin 2\alpha + \cos^2\alpha]$$

$$\varphi_{rs} = PM\Big\{2a(\alpha_2\beta_1 - \alpha_1\beta_2)\sin^2\alpha + [a(\beta_1 - \beta_2) + b\left(\alpha_1\alpha_2 - \alpha_2^2 + \beta_1\beta_2 - \beta_2^2\right)]\sin 2\alpha$$
$$+ 2b(\alpha_1 - \alpha_2)\cos^2\alpha\Big\} - PNb\alpha_1[\left(\alpha_2^2 + \beta_2^2\right)\sin^2\alpha + \alpha_2 \sin 2\alpha + \cos^2\alpha]$$

$$\varphi_{ic} = -PM\Big\{2a(\alpha_2\beta_1 - \alpha_1\beta_2)\sin^2\alpha + [a(\beta_1 - \beta_2) + b\left(\alpha_1\alpha_2 - \alpha_2^2 + \beta_1\beta_2 - \beta_2^2\right)]\sin 2\alpha$$
$$+ 2b(\alpha_1 - \alpha_2)\cos^2\alpha\Big\}$$

$$\varphi_{is} = PM\Big\{2a\left(\alpha_1\alpha_2 - \alpha_2^2 + \beta_1\beta_2 - \beta_2^2\right)\sin^2\alpha + \left[a(\alpha_2 - \alpha_1) + b(\alpha_1\beta_2 - \alpha_2\beta_1)\right]\sin 2\alpha$$
$$+ 2b(\beta_2 - \beta_1)\cos^2\alpha\Big\} - PNb\beta_1[\left(\alpha_2^2 + \beta_2^2\right)\sin^2\alpha + \alpha_2 \sin 2\alpha + \cos^2\alpha]$$

$$\phi_{rc} = PM\Big\{2a\big(\alpha_1\alpha_2 - \alpha_1^2 + \beta_1\beta_2 - \beta_1^2\big)\sin^2\alpha + \big[a(\alpha_2 - \alpha_1) + b\big(\alpha_2\beta_1 - \alpha_1\beta_2\big)\big]\sin 2\alpha$$
$$+ 2b\big(\beta_1 - \beta_2\big)\cos^2\alpha\Big\} + PNa\big[\big(\alpha_1^2 - \beta_1^2 - 2\alpha_1\alpha_2\big)\sin^2\alpha - \alpha_2\sin 2\alpha - \cos^2\alpha\big]$$

$$\phi_{rs} = PM\Big\{2a\big(\alpha_1\beta_2 - \alpha_2\beta_1\big)\sin^2\alpha + [a\big(\beta_2 - \beta_1\big) + b\big(\alpha_1\alpha_2 - \alpha_1^2 + \beta_1\beta_2 - \beta_1^2\big)]\sin 2\alpha$$
$$+ 2b(\alpha_2 - \alpha_1)\cos^2\alpha\Big\} + PN\frac{b}{2}\Big\{2\alpha_1\big(\alpha_2^2 + \beta_2^2\big)\sin^2\alpha + \big(\alpha_1^2 + \alpha_2^2 - \beta_1^2 + \beta_2^2\big)\sin 2\alpha + 2\alpha_1\cos^2\alpha\Big\}$$

$$\phi_{ic} = -PM\Big\{2a\big(\alpha_1\beta_2 - \alpha_2\beta_1\big)\sin^2\alpha + [a\big(\beta_2 - \beta_1\big) + b\big(\alpha_1\alpha_2 - \alpha_1^2 + \beta_1\beta_2 - \beta_1^2\big)]\sin 2\alpha$$
$$+ 2b(\alpha_2 - \alpha_1)\cos^2\alpha\Big\} + PN\frac{a}{2\beta_2}\Big\{2\Big[\alpha_2\big(\alpha_1^2 - \beta_1^2\big) - \alpha_1\big(\alpha_2^2 - \beta_2^2\big)\Big]\sin^2\alpha$$
$$+ \big(\alpha_1^2 - \alpha_2^2 + \beta_2^2 - \beta_1^2\big)\sin 2\alpha + 2(\alpha_1 - \alpha_2)\cos^2\alpha\Big\}$$

$$\phi_{is} = PM\Big\{2a\big(\alpha_1\alpha_2 - \alpha_1^2 + \beta_1\beta_2 - \beta_1^2\big)\sin^2\alpha + \big[a(\alpha_2 - \alpha_1) + b\big(\alpha_2\beta_1 - \alpha_1\beta_2\big)\big]\sin 2\alpha$$
$$+ 2b\big(\beta_1 - \beta_2\big)\cos^2\alpha\Big\} + PN\frac{b}{2\beta_2}[2\big(\alpha_2^2 + \beta_2^2\big)\big(-\alpha_1^2 + \alpha_1\alpha_2 + \beta_1^2\big)\sin^2\alpha$$
$$+ \alpha_2\big(-\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2\big)\sin 2\alpha + 2\big(\alpha_2^2 - \alpha_1\alpha_2 + \beta_2^2\big)\cos^2\alpha]$$

where $M$ and $N$ are constants, given by

$$M = \frac{1}{4}\Big[(\alpha_2 - \alpha_1)^2 + \big(\beta_2 - \beta_1\big)^2\Big]^{-1}, \qquad N = \frac{1}{4}\Big[(\alpha_2 - \alpha_1)^2 + \beta_2^2 - \beta_1^2\Big]^{-1}$$

It follows from formulas (14) that the displacements of the boundary of the hole are

$$u_1 = 2\big(\varphi_{rc}p_{r1} + \phi_{rc}p_{r2} - \varphi_{ic}p_{i1} - \phi_{ic}p_{i2}\big)\cos\theta + 2\big(\varphi_{rs}p_{r1} + \phi_{rs}p_{r2} - \varphi_{is}p_{i1} - \phi_{is}p_{i2}\big)\sin\theta$$

$$u_2 = 2\big(\varphi_{rc}q_{r1} + \phi_{rc}q_{r2} - \varphi_{ic}q_{i1} - \phi_{ic}q_{i2}\big)\cos\theta + 2\big(\varphi_{rs}q_{r1} + \phi_{rs}q_{r2} - \varphi_{is}q_{i1} - \phi_{is}q_{i2}\big)\sin\theta \qquad \text{(A3)}$$

where $p_{rk}$, $q_{rk}$ and $p_{ik}$, $q_{ik}$ are real and imaginary parts of $p_k$ and $q_k$, correspondingly.

In $\zeta$-plane, $n_1(\theta)\,\mathrm{d}\Gamma = -b\cos\theta\,\mathrm{d}\theta$ and $n_2(\theta)\,\mathrm{d}\Gamma = a\sin\theta\,\mathrm{d}\theta$, so that integral (4) can be readily evaluated:

$$\Delta\varepsilon_{11} = \frac{2\pi b}{A}\big(\varphi_{rc}p_{r1} + \phi_{rc}p_{r2} - \varphi_{ic}p_{i1} - \phi_{ic}p_{i2}\big)$$

$$\Delta\varepsilon_{12} = \frac{\pi a}{A}\big(-\varphi_{rs}p_{r1} - \phi_{rs}p_{r2} + \varphi_{is}p_{i1} + \phi_{is}p_{i2}\big) + \frac{\pi b}{A}\big(\varphi_{rc}q_{r1} + \phi_{rc}q_{r2} - \varphi_{ic}q_{i1} - \phi_{ic}q_{i2}\big)$$

$$\Delta\varepsilon_{22} = \frac{2\pi a}{A}\left( - \varphi_{rs} q_{r1} - \phi_{rs} q_{r2} + \varphi_{is} q_{i1} + \phi_{is} q_{i2}\right) \tag{A4}$$

Comparing with the general structure of the relation between $\Delta\varepsilon$ and loading $P$ (components of (5) in the case of uniaxial loading):

$$\Delta\varepsilon_{11} = P\left(H_{1111}\cos^2\alpha + H_{1112}\sin 2\alpha + H_{1122}\sin^2\alpha\right)$$

$$\Delta\varepsilon_{12} = P\left(H_{1112}\cos^2\alpha + H_{1212}\sin 2\alpha + H_{1222}\sin^2\alpha\right)$$

$$\Delta\varepsilon_{22} = P\left(H_{1122}\cos^2\alpha + H_{1222}\sin 2\alpha + H_{2222}\sin^2\alpha\right) \tag{A5}$$

and equating coefficients at $\cos^2\alpha$, $\sin 2\alpha$ and $\sin^2\alpha$ in Eqs. (A4) and (A5), yields expressions (20) for components of **H**-tensor.

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